ad-NILPOTENT IDEALS CONTAINING A FIXED NUMBER OF SIMPLE ROOT SPACES

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Abstract. We give formulas for the number of ad-nilpotent ideals of a Borel subalgebra of a Lie algebra of type B or D containing a fixed number of root spaces attached to simple roots. This result solves positively a conjecture of Panyushev [12, 3.5] and affords a complete knowledge of the above statistics for any simple Lie algebra. We also study the restriction of the above statistics to the abelian ideals of a Borel subalgebra, obtaining uniform results for any simple Lie algebra.

1. Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. Fix a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and let $\mathfrak{n}$ be its nilradical. If $\mathfrak{g}$ is of type X, denote by $\mathcal{I}(X)$ denote the set of ad-nilpotent ideals $\mathfrak{b}$, i.e. the ideals of $\mathfrak{b}$ which are contained in $\mathfrak{n}$. Let Δ^+, Π denote respectively the positive and simple systems of the root system Δ of $\mathfrak{g}$ corresponding to $\mathfrak{b}$. Then $\mathfrak{i} \in \mathcal{I}(X)$ if and only if $\mathfrak{i} = \bigoplus\limits_{\alpha\in\Phi_{\mathfrak{i}}} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space attached to α and $\Phi_{\mathfrak{i}} \subseteq \Delta^+$ is a dual order ideal of Δ^+ (w.r.t. the usual order: $\alpha < \beta$ is $\beta - \alpha$ is a sum of positive roots). ad-nilpotent ideals have been intensively investigated in recent literature: see references in [12]. The first goal of this short paper is to solve positively conjecture 3.5 of [12]. This conjecture regards the following statistics on $\mathcal{I}(X)$:

$$P_X(j) = |\,\{\mathfrak{i} \in \mathcal{I}(X) : |\Pi \cap \Phi_{\mathfrak{i}}| = j\}\,|$$

($0 \leq j \leq n$). The formulas expressing $P_X(j)$ for the classical Lie algebras are given in the following theorem. The result in type A has been proved in [12, Theorem 3.4], together with the equality $P_{B_n} = P_{C_n}$. The formulas for types B, D are conjecture 3.5 of the same paper.

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Theorem 1.1. *For $0 \leq j \leq n$ we have*

$$P_{A_n}(j) = \frac{j+1}{n+1}\binom{2n-j}{n},$$

$$P_{B_n}(j) = P_{C_n}(j) = \binom{2n-j-1}{n-1},$$

$$P_{D_n}(j) = \begin{cases} \binom{2n-2}{n-2} + \binom{2n-3}{n-3} & \text{if } j = 0 \\ \binom{2n-2-j}{n-2} + \binom{2n-3-j}{n-2} & \text{if } 1 \leq j \leq n. \end{cases}$$

We remark that the numerical values of $P_X(j)$ in the exceptional cases are easily calculated from the knowledge of $P_X(0)$ using the inclusion-exclusion principle: see [12, §3]. On the other hand, the number $P_X(0)$ can be uniformly described: see Remark 2.1.

The relevance of the statistics P_X is motivated by the following discussion. It is known [4] that the cardinality of $\mathcal{I}$ is given by the generalized Catalan number $\frac{1}{|W|}\prod_{i=1}^{n}(e_i+h+1)$ (see Remark 2.1 for undefined notation) as well as that of *clusters*, certain subsets of $\Delta^+ \cup -\Pi$ which play a major role in Zelevinsky's theory of cluster algebras [7]. Panyushev noticed that $P_X(j)$ also counts the number of clusters having j elements in $-\Pi$. Looking for a conceptual explanation of the interplay between *ad*-nilpotent ideals and clusters is an interesting open problem.

Theorem 1.1 is proved in the next section. The final section deals with a formula for the same statistics on the subset $\mathcal{I}^{ab}$ of $\mathcal{I}$ consisting of abelian ideals. The study of $\mathcal{I}^{ab}$, pursued by Kostant, started an intense research activity which was later extended by considering *ad*-nilpotent ideals. Abelian ideals turn out to appear in several contexts, ranging from the structure of the exterior algebra of $\mathfrak{g}$ [9], to affine algebras [2] and to difficult problems in classical invariant theory [11]. The key fact originating this activity is the following celebrated enumerative result by Dale Peterson, which we are going to exploit:

$$|\mathcal{I}^{ab}| = 2^{rk(\mathfrak{g})}. \tag{1.1}$$

Regarding our statistics, we obtain the following "uniform" result. Let P, Q denote the weight and root lattice of Δ and let $z(\mathfrak{g}) = |P/Q|$ be the connection index.

Theorem 1.2. *The number $P_X^{ab}(j)$ of abelian ideals of $\mathfrak{b}$ in a Lie algebra $\mathfrak{g}$ of type X and rank n containing j simple roots is given by*

$$P_X^{ab}(j) = \begin{cases} 2^n - z(\mathfrak{g}) + 1 & \text{if } j=0, \\ z(\mathfrak{g}) - 1 & \text{if } j=1, \\ 0 & \text{if } j > 1. \end{cases}$$

2. Proof of Theorem 1.1

Our approach to Panyushev's conjecture is based on Shi's encoding [13] of ad-nilpotent ideals for classical Lie algebras via (possibly shifted) shapes as formulated in [3] . More precisely, consider a staircase diagram T_X of shape $(n, n-1, \ldots, 1)$ in type A_n (respectively a shifted staircase diagram of shape $(2n-1, 2n-3, \ldots, 1)$ for B_n and C_n, and of shape $(2n-2, 2n-4, \ldots, 2)$ for D_n). Arrange in the diagram the positive roots of Δ according to the formulas

$$\tau_{i,j} = \alpha_i + \cdots + \alpha_{n-j+1} \qquad 1 \le i \le j \le n.$$

$$\tau_{i,j} = \begin{cases} \alpha_i + \cdots + \alpha_{j-1} + 2(\alpha_j + \cdots + \alpha_{n-1}) + \alpha_n & \text{if } j \le n-1, \\ \alpha_i + \cdots + \alpha_{2n-j} & \text{if } n \le j \le 2n-i. \end{cases}$$

$$\tau_{i,j} = \begin{cases} \alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_n) & \text{if } j \le n-1, \\ \alpha_i + \cdots + \alpha_{2n-j} & \text{if } n \le j \le 2n-i. \end{cases}$$

$$\tau_{i,j} = \begin{cases} \alpha_i + \cdots + \alpha_j + 2(\alpha_{j+1} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n & \text{if } j \le n-2, \\ \alpha_i + \cdots + \alpha_{n-2} + \alpha_n & \text{if } j = n-1, \\ \alpha_i + \cdots + \alpha_{2n-j-1} & \text{if } n \le j \le 2n-1-i. \end{cases}$$

in types A_n, C_n, B_n, D_n respectively. E.g., in types A_4, C_3, B_3, D_4 we have, respectively

$$\begin{array}{llll}
\alpha_1+\alpha_2+\alpha_3+\alpha_4 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\
\alpha_2+\alpha_3+\alpha_4 & \alpha_2+\alpha_3 & \alpha_2 & \\
\alpha_3+\alpha_4 & \alpha_3 & & \\
\alpha_4 & & &
\end{array}$$

$$\begin{array}{lllll}
2\alpha_1+2\alpha_2+\alpha_3 & \alpha_1+2\alpha_2+\alpha_3 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\
 & 2\alpha_2+\alpha_3 & \alpha_2+\alpha_3 & \alpha_2 & \\
 & & \alpha_3 & &
\end{array}$$

$$\begin{array}{lllll}
\alpha_1+2\alpha_2+2\alpha_3 & \alpha_1+\alpha_2+2\alpha_3 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\
 & \alpha_2+2\alpha_3 & \alpha_2+\alpha_3 & \alpha_2 & \\
 & & \alpha_3 & &
\end{array}$$

$$\begin{array}{llllll}
\alpha_1+2\alpha_2+\alpha_3+\alpha_4 & \alpha_1+\alpha_2+\alpha_3+\alpha_4 & \alpha_1+\alpha_2+\alpha_4 & \alpha_1+\alpha_2+\alpha_3 & \alpha_1+\alpha_2 & \alpha_1 \\
 & \alpha_2+\alpha_3+\alpha_4 & \alpha_2+\alpha_4 & \alpha_2+\alpha_3 & \alpha_2 & \\
 & & \alpha_4 & \alpha_3 & &
\end{array}$$

Then $\mathcal{I}(X)$ is in bijection with the set $\mathcal{S}_X$ of subdiagrams of T_X when $X = A, B, C$ whereas in type D one has to consider also the sets of boxes of T_D which become subdiagrams of T_D upon switching columns $n-1, n$ (see [13] or [3]).

In turn to each subdiagram we can associate a lattice path of length $2n$, starting from the origin and never going under the x-axis, with step vectors $(1,1)$, $(1,-1)$ (see [10]). The correspondence between subdiagrams and paths is best explained with an example at hand. Let $n = 9$ and consider, for type B_n or C_n, the shifted partition $(16, 13, 11, 8, 7, 5, 3)$, see Figure 1 (here, as in Figure 3, the origin coincides with the left upper corner of the diagram, and the y-axis points downwards). Connect the point $(2n, 0)$ to the border of the subdiagram with an horizontal segment, and consider the zig-zag line formed by the horizontal segment and the right border of the subdiagram. Rotate the figure by 45° in the positive direction and then flip it across a vertical line. After rescaling (in the obvious way) we obtain the desired lattice path. See Figure 2 for the path corresponding to the partition of Figure 1. (To make a comparison easy, the steps which correspond to thick segments in Figure 1 are also made thick in Figure 2.)

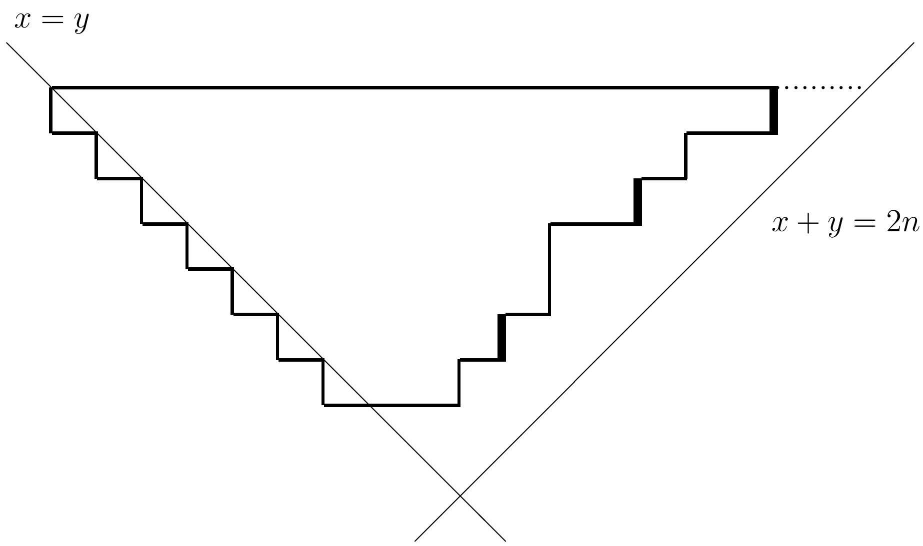

Figure 1

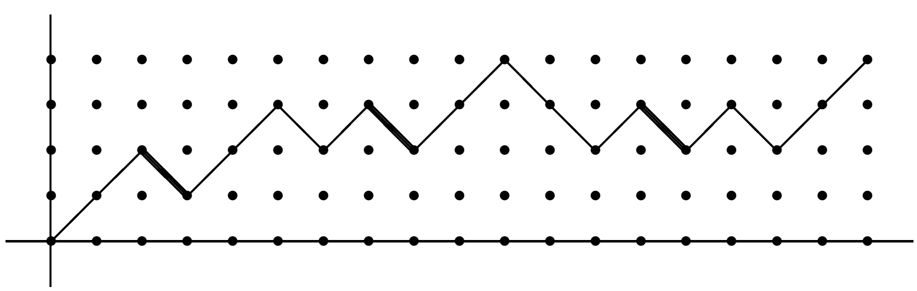

Figure 2

So we have associated to any subdiagram of T_{B_n} (or T_{C_n}) a lattice path of length $2n$. In a similar way we can associate to any subdiagram of T_{D_n} a lattice path of length $2n-1$. Slight modifications are needed to define a correspondence in type A_n. Start from the point $(n+1, 0)$, reach and follow the right border of the diagram. End in the point

$(0, n+1)$: see Figures 3,4 for the case of the partition $(5,3,1,1,1,0,0)$, relative to A_7.

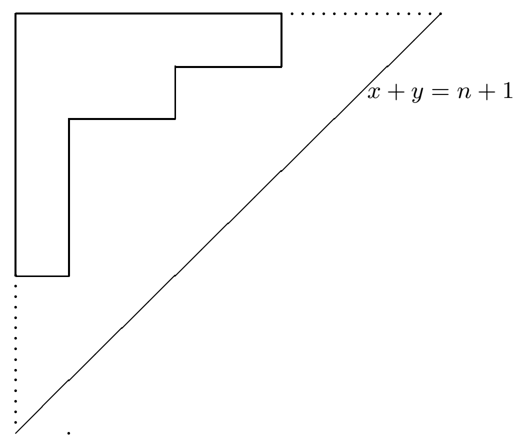

Figure 3

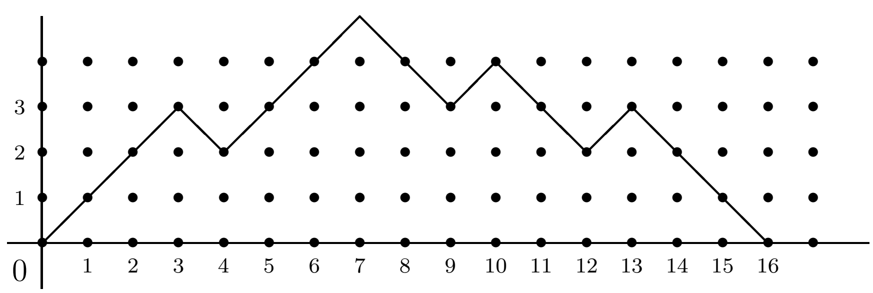

Figure 4

In type A_n this correspondence turns out to be a bijection between $\mathcal{I}(A_n)$ and the set of Dyck paths of length $2n+2$, whereas in types B_n, C_n one gets a bijection with the set of paths of length $2n$ not necessarily ending on the x-axis.

Remark that in cases B_n, C_n our statistics P_X translates into the one which counts the number of *returns* of the paths, i.e. the number of contact points of the path with the x-axis minus one. In type A_n the statistics P_X counts the number of returns minus one (so the statistics has value 0 for the path of Figure 4).

Denote by $\mathcal{B}_{n,h,j}$ the set of paths of the previous type having length n, ending in the point (n,h) and having exactly j returns. The enumeration of such objects has been known since a long time (see [8, §2] for historical details and generalizations). As usual we set $\binom{n}{m} = 0$ if $m < 0$.

Proposition 2.1. [6, 13, Cor. 3.2] *Assume $n \equiv h$, mod 2. Then*

$$|\mathcal{B}_{n,h,j}| = \binom{n-(j+1)}{\frac{n+h}{2}-1} - \binom{n-(j+1)}{\frac{n+h}{2}}. \tag{2.1}$$

Note that if a path has length n and ends at height j, then $n+j$ is even. In particular, if $n+h$ is odd then $\mathcal{B}_{n,h,j}=\emptyset$ for any j. We have immediately

$$P_{A_n}(j)=|\mathcal{B}_{2n+2,0,j+1}|=\frac{j+1}{n+1}\binom{2n-j}{n},$$

$$P_{B_n}(j)=P_{C_n}(j)=\sum_{h=0}^{2n}|\mathcal{B}_{2n,h,j}|=\binom{2n-j-1}{n-1}$$

which are the desired formulas in cases A_n, B_n, C_n.

For type D we argue as follows. First observe that, in the diagramatic encoding, ideals can be counted as

$$2|\mathcal{S}_{D_n}|-|\mathcal{D}_n| \tag{2.2}$$

$\mathcal{D}_n$ being the set of subdiagrams of T_{D_n} having columns $n-1,n$ of equal length. So we have to understand our statistics on $\mathcal{S}_{D_n}$ and on $\mathcal{D}_n$. Ideals corresponding to subdiagrams in $\mathcal{S}_{D_n}$ give rise to paths starting from the origin and having length $2n-1$. The number of simple roots belonging to $\Phi_\mathfrak{i}$ for such an ideal $\mathfrak{i}$ is exactly the number of returns of the corresponding path precisely when the ideal does not contain α_n. In this latter case to get the number of simple roots one has to add 1 to the number of returns. On the other hand the ideals containing α_n are exactly the ones giving rise to paths ending at height 1. Therefore the piece in degree j of our statistics coming from $\mathcal{S}_{D_n}$ is

$$\begin{aligned}&\sum_{h=3}^{2n-1}|\mathcal{B}_{2n-1,h,j}|+|\mathcal{B}_{2n-1,1,j-1}|\\&=\binom{2n-j-2}{n}+\binom{2n-j-1}{n-1}-\binom{2n-j-1}{n}\\&=\binom{2n-j-2}{n-2}.\end{aligned}$$

We have used relation (2.1) to evaluate the left hand side of the previous expression.

Now remark that the contribution to the piece of degree j of our statistics coming from $\mathcal{D}_n$ is

$$P_{B_{n-1}}(j)-P_{A_{n-2}}(j-1)+P_{A_{n-2}}(j-2).$$

Note in fact that to any diagram in $\mathcal{D}_n$ we can associate a diagram in $T_{B_{n-1}}$ by deleting the n-th column. In so doing our statistics counts:
(a) all paths for type B_{n-1} having j returns and end point not lying on the x-axis;
(b) all paths for type B_{n-1} having $j-1$ returns and end point on the x-axis.

It is clear that paths for B_{n-1} having k returns and end point on the x-axis are the same as paths for A_{n-2} with $k-1$ returns. Hence

contribution (a) is $P_{B_{n-1}}(j) - P_{A_{n-2}}(j-1)$, and contribution (b) is $P_{A_{n-2}}(j-2)$. Relation (2.2) and some elementary calculations yield the last formula in the Theorem.

Remark 2.1. It is worth recalling that the value $P_X(0)$ has a special geometric meaning. Indeed, ad-nilpotent ideals correspond to connected components in the dominant chamber of $\mathfrak{h}_{\mathbb{R}}$ ($\mathfrak{h}$ being a Cartan subalgebra of $\mathfrak{g}$) determined by the hyperplanes $(\alpha, x) = 0$, $(\alpha, x) = 1$, $\alpha \in \Delta^+$. More precisely, the open region associated to the ideal $\mathfrak{i}$ is determined by the inequalities $0 < (\alpha, x) < 1$ if $\mathfrak{g}_\alpha \not\subset \mathfrak{i}$, and $(\alpha, x) > 1$ if $\mathfrak{g}_\alpha \subset \mathfrak{i}$. Panyushev proved that an ideal in $\mathcal{I}$ does not contain a simple root space if and only if the corresponding region is bounded (see [12, Proposition 3.7]). He also found the following remarkable formula (see [12, Proposition 3.10]):

$$P_X(0) = \frac{1}{|W|}\prod_{i=1}^{n}(h + e_i - 1).$$

Here W is the Weyl group, h the Coxeter number and $e_1, \dots, e_n$ the exponents of $\mathfrak{g}$. $P_X(0)$ is also the number of positive clusters.

3. Proof of Theorem 1.2

Lemma 3.1. *An abelian ideal* $\mathfrak{i} \in \mathcal{I}^{ab}$ *may contain at most one simple root space.*

Proof. Let $\alpha, \alpha' \in \Pi$ such that $\mathfrak{g}_\alpha, \mathfrak{g}_{\alpha'} \subset \mathfrak{i}$. Consider a minimal length path from α to α' in the Dynkin diagram of $\mathfrak{g}$. By Corollaire 3 in [1, VI, 1.7] the sum γ of the simple roots in the path belongs to Δ^+ as well as $\gamma - \alpha$. Moreover $\gamma > \alpha$, $\gamma - \alpha > \alpha'$. Therefore $\mathfrak{g}_\gamma \subset \mathfrak{i}$, $\mathfrak{g}_{\gamma-\alpha} \subset \mathfrak{i}$. But $[\mathfrak{g}_\alpha, \mathfrak{g}_{\gamma-\alpha}] = \mathfrak{g}_\gamma$, hence $\mathfrak{i}$ is not abelian. □

Recall that an ad-nilpotent ideal is nilpotent, i.e. its descending central series

$$\mathfrak{i} \supset [\mathfrak{i}, \mathfrak{i}] \supset [[\mathfrak{i}, \mathfrak{i}], \mathfrak{i}] \supset [[[\mathfrak{i}, \mathfrak{i}], \mathfrak{i}]\, \mathfrak{i}] \supset \cdots\cdots$$

has a finite number $n(\mathfrak{i})$ of non zero terms. In particular, $\mathfrak{i}$ is an abelian ideal if and only if $n(\mathfrak{i}) \leq 1$. Also recall that ad-nilpotent ideals are in canonical bijection with antichains (i.e., subset formed by mutually non-comparable elements) in the root poset. The correspondence is given by mapping an ideal to its minimal roots w.r.t $<$, and the inverse map associates to an antichain A the ideal $\bigoplus\limits_{\beta \in A} \bigoplus\limits_{\alpha \geq \beta} \mathfrak{g}_\alpha$.

If $\Pi = \{\alpha_1, \dots, \alpha_n\}$, denote by $\theta = \sum\limits_{i=1}^{n} a_i \alpha_i$ the highest root of Δ.

Lemma 3.2. *Let* $\mathfrak{i}_j = \bigoplus\limits_{\beta \geq \alpha_j} \mathfrak{g}_\alpha$, $1 \leq j \leq n$. *Then*

$$n(\mathfrak{i}_j) = a_j.$$

Proof. We use the following result of Chari, Dolbin and Ridenour [5, Theorem 1]. Let $\mathfrak{i}$ an *ad*-nilpotent ideal corresponding to the antichain $A = \{\beta_1, \dots, \beta_k\}$. Then $n(\mathfrak{i}) = s$ if and only if s is the minimal non-negative integer such that $\beta_{i_1} + \ldots + \beta_{i_{s+1}} \not\leq \theta$ (repetitions in the β are allowed). The claim follows immediately, because the antichain attached to $\mathfrak{i}_j$ consists only of α_j, and $\theta - a_j\alpha_j = \sum_{i=1}^{j-1} a_i\alpha_i + \sum_{i=j+1}^{n} a_i\alpha_i$ belongs to the positive root lattice, whereas

$$\theta - (a_j + 1)\alpha_j = \sum_{i=1}^{j-1} a_i\alpha_i - \alpha_j + \sum_{i=j+1}^{n} a_i\alpha_i$$

does not. □

We are ready to prove Theorem 1.2. The result follows combining (1.1) and Lemma 3.1 if we prove that $P_X^{ab}(1) = z(\mathfrak{g}) - 1$. On the other hand Lemma 3.2 implies that $P_X^{ab}(1)$ equals the number of indices i such that $a_i = 1$. The latter number is known to coincide with $z(\mathfrak{g}) - 1$ (see [1, VI, §2.3]).

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